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A Positivity Result Applied to Difference Equations

DARREL HANKERSON

Department of Algebra, Combinatorics and Analysis, Auburn University, Auburn, Alabama 36849-5307, U.S.A.

AND

ALLAN PETERSON

Department of Mathematics and Statistics, University of Nebraska, Lincoln, Nebraska 68588-0323, U.S.A.

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We will prove comparison theorems for the least positive eigenvalues of (1), (3) and (2), (3) below. Consider

$$(-1)^{n-k}\, Ly(t) = \lambda P(t)\, y(t+k), \tag{1}$$

$$(-1)^{n-k}\, Ly(t) = \Lambda Q(t)\, y(t+k) \tag{2}$$

with boundary conditions

$$\begin{aligned} \Delta^i y(a) &= 0, \qquad 0 \leqslant i \leqslant k-1, \\ \Delta^i y(b+k+1) &= 0, \qquad 0 \leqslant i \leqslant n-k-1, \end{aligned} \tag{3}$$

where a and b $(>a)$ are integers and t is a discrete variable. Here $P(t)$ and $Q(t)$ are $m \times m$ matrix functions defined for $t \in [a, b]$. Further, $k \in \{1, ..., n-1\}$, λ, Λ are scalar parameters, Δ is the difference operator defined by $\Delta y(t) = y(t+1) - y(t)$, and a solution $y(t)$ of (1) (or (2)) is an m-dimensional vector function defined on $[a, b+n]$. $Ly(t) = 0$ is the nth order difference equation defined by

$$Ly(t) = \sum_{i=0}^{n} \alpha_i(t)\, y(t+i) = 0, \tag{4}$$

where the coefficients are scalar functions defined on $[a, b]$ with $\alpha_n(t) \equiv 1$ and

$$(-1)^n\, \alpha_0(t) > 0 \tag{5}$$

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for $t \in [a, b]$. In [2, Chap. XIV], Fort considers (1), (3) with $n=2$ and $k=1$.

Let x be a scalar and let $Lx=0$ denote the scalar equation corresponding to (4) defined by

$$Lx(t) = \sum_{i=0}^{n} \alpha_i(t)\, x(t+i) = 0.$$

We say that a solution $x(t)$ of $Lx=0$ has a *generalized zero* at t_0 in case either $x(t_0)=0$ or there exists an integer j with $1 \leqslant j \leqslant t_0 - a$ such that $(-1)^j\, x(t_0-j)\, x(t_0) > 0$ and if $j>1$, $x(t)=0$, $t_0-j<t<t_0$. Hypothesis (5) guarantees (see [11]) that a nontrivial solution of $Lx=0$ cannot have $n-1$ zeros at $t, ..., t+n-2$ and a generalized zero at $t+n-1$.

We say $Lx=0$ is *right* $(j, n-j)$*-disconjugate* on $[a, b+n]$ provided there is no nontrivial solution $x(t)$ and integers α, β, with $a \leqslant \alpha < \alpha + j \leqslant \beta \leqslant b+j+1$, such that

$$x(\alpha+i)=0, \qquad 0 \leqslant i \leqslant j-1$$

$$x(\beta+i)=0, \qquad 0 \leqslant i \leqslant n-j-2$$

and x has a generalized zero at $\beta+n-j-1$. We say $Lx=0$ is *left* $(j, n-j)$*-disconjugate* on $[a, b+n]$ provided there is no nontrivial solution $x(t)$ and integers α, β, with $a \leqslant \alpha < \alpha + j \leqslant \beta \leqslant b+j+1$, such that

$$x(\alpha+i)=0, \qquad 0 \leqslant i \leqslant j-2$$

$$x(\beta+i)=0, \qquad 0 \leqslant i \leqslant n-j-1$$

and x has a generalized zero at $\alpha+j-1$. If $Lx=0$ is left and right $(j, n-j)$-disconjugate on $[a, b+n]$, then we say that $Lx=0$ is $(j, n-j)$*-disconjugate* on $[a, b+n]$. $Lx=0$ is *disconjugate* (see [6]) on $[a, b+n]$ provided no nontrivial solution has n generalized zeros on $[a, b+n]$. It is known that if $Lx=0$ is right $(j, n-j)$-disconjugate on $[a, b+n]$, $1 \leqslant j \leqslant n-1$, then $Lx=0$ is disconjugate on $[a, b+n]$.

If $x_1(t), ..., x_j(t)$ are solutions of $Lx=0$, then we define the Wronskian of $x_1(t), ..., x_j(t)$ by

$$W[x_1(t), ..., x_j(t)] = \begin{vmatrix} x_1(t) & \cdots & x_j(t) \\ \Delta x_1(t) & \cdots & \Delta x_j(t) \\ \vdots & \ddots & \vdots \\ \Delta^{j-1} x_1(t) & \cdots & \Delta^{j-1} x_j(t) \end{vmatrix}$$

$$= \begin{vmatrix} x_1(t) & \cdots & x_j(t) \\ x_1(t+1) & \cdots & x_j(t+1) \\ \vdots & \ddots & \vdots \\ x_1(t+j-1) & \cdots & x_j(t+j-1) \end{vmatrix}.$$

Let $u_j(t, s)$, $0 \leqslant j \leqslant n-1$, be solutions of $Lx = 0$ satisfying the partial set of initial conditions

$$\Delta^i u_j(s, s) = \delta_{ij}, \qquad 0 \leqslant i \leqslant j,$$

where δ_{ij} is the Kronecker delta. It was shown in [13] that $Lx = 0$ is right $(j, n-j)$-disconjugate on $[a, b+n]$ if and only if

$$W[u_j(t, s), ..., u_{n-1}(t, s)] > 0, \qquad a \leqslant s \leqslant t - j \leqslant b + 1.$$

This is the first hint of a positivity result.

We make the following assumption throughout this paper concerning the equation $Lx = 0$:

> Either $Lx = 0$ is disconjugate on $[a, b+n]$, or $2 \leqslant k \leqslant n-1$ and $Lx = 0$ is $(j, n-j)$-disconjugate on $[a+k-j, b+n+k-j]$ for $k - 1 \leqslant j \leqslant n-1$. (H)

Our results appear to be new even when $Lx = 0$ is disconjugate on $[a, b+n]$.

We now state the positivity result that we will use later. For ease of reference we call it Theorem 1. For a proof of this result, see [6] when $Lx = 0$ is disconjugate on $[a, b+n]$, and see [12] if the second condition in (H) holds.

THEOREM 1. *If* (H) *holds, then the Green's function* $G(t, s)$ *for the* $(k, n-k)$*-boundary value problem*

$$(-1)^{n-k} Lx(t) = h(t)$$

$$\Delta^i x(a) = 0, \qquad 0 \leqslant i \leqslant k-1$$

$$\Delta^i x(b+k+1) = 0, \qquad 0 \leqslant i \leqslant n-k-1$$

satisfies

$$G(t, s) > 0, \qquad t \in [a+k, b+k], \; s \in [a, b].$$

The other main tool that we will use is cone theory in a Banach space as developed by Krasnosel'skiĭ. For applications of this cone theory see [3–5, 7, 8, 10, 14–17]. We now introduce the relevant cone theory that we use in this paper.

Let $\mathscr{B}$ be a Banach space. A closed nonempty subset $\mathscr{P}$ of $\mathscr{B}$ is called a *cone* provided that whenever $u, v \in \mathscr{P}$ it follows that $\alpha u + \beta v \in \mathscr{P}$ for all $\alpha \geqslant 0$, $\beta \geqslant 0$, and whenever $u, -u \in \mathscr{P}$, then $u = 0$. We say that a cone $\mathscr{P}$ is *reproducing* provided $\mathscr{B} = \mathscr{P} - \mathscr{P} \equiv \{u - v: u, v \in \mathscr{P}\}$. We write $u \leqslant v$

provided $v-u\in\mathscr{P}$. If M and N are operators on $\mathscr{B}$, then we write $M\leqslant N$ (with respect to $\mathscr{P}$) provided $Mu\leqslant Nu$ for all $u\in\mathscr{P}$. A bounded linear operator M is *u_0-positive* provided $u_0\in\mathscr{P}$ and for each nonzero $u\in\mathscr{P}$, there are positive numbers k_1, k_2 (which in general depend on u) such that $k_1u_0\leqslant Mu\leqslant k_2u_0$.

We will use the following results from cone theory which we state here for easy reference. The first two appear in [9], and the third result appears in [17].

THEOREM 2. *Assume $\mathscr{P}$ is a reproducing cone and M is a linear compact operator which leaves the cone $\mathscr{P}$ invariant. Assume there is a nontrivial $u_0\in\mathscr{B}$ and an $\varepsilon_0>0$ such that $Mu_0\geqslant\varepsilon_0u_0$. Then M has at least one eigenvector $z_0\in\mathscr{P}$ with corresponding eigenvalue $\lambda_0\geqslant\varepsilon_0$ such that λ_0 is an upper bound for the moduli of the eigenvalues of M.*

THEOREM 3. *Assume $\mathscr{P}$ is a reproducing cone and M is a compact u_0-positive linear operator. Then M has an essentially unique eigenvector in $\mathscr{P}$ and the corresponding eigenvalue is simple, positive, and larger than the modulus of any other eigenvalue of M.*

THEOREM 4. *Assume M and N are linear operators and that at least one of them is u_0-positive. If $M\leqslant N$ and there exist nontrivial u_1, $u_2\in\mathscr{P}$, $\lambda_1,\lambda_2>0$ such that $Mu_1\geqslant\lambda_1u_1$ and $Nu_2\leqslant\lambda_2u_2$, then $\lambda_1\leqslant\lambda_2$ and if $\lambda_1=\lambda_2$ then u_1 is a scalar multiple of u_2.*

The Banach space that we are interested in here is

$$\mathscr{B}=\{y\colon [a,b+n]\to R^m \mid \Delta^iy(a)=0, 0\leqslant i\leqslant k-1,$$
$$\Delta^iy(b+k+1)=0, 0\leqslant i\leqslant n-k-1\},$$

where the norm on $\mathscr{B}$ is defined by $\|y\|=\max\{|y(t)| : t\in[a+k,b+k]\}$ and $|\cdot|$ is the Euclidean norm. Let $\mathscr{K}$ be a reproducing cone in R^m and define the cone $\mathscr{P}$ by

$$\mathscr{P}=\{y\in\mathscr{B}\colon y(t)\in\mathscr{K}, t\in[a+k,b+k]\}.$$

It is easy to show that $\mathscr{P}$ is a reproducing cone.

Define operators M and N on $\mathscr{B}$ by

$$Mu(t)=\sum_{s=a}^{b} G(t,s)\,P(s)\,u(s+k)$$

$$Nu(t)=\sum_{s=a}^{b} G(t,s)\,Q(s)\,u(s+k)$$

for $t \in [a, b+n]$. It can be shown that M and N are compact linear operators.

Note that if $u \in \mathscr{B}$ and $h(t) = Mu(t)$, then $h(t)$ is a solution of

$$(-1)^{n-k} Lh(t) = P(t)\, u(t+k)$$
$$\Delta^i h(a) = 0, \qquad 0 \leqslant i \leqslant k-1$$
$$\Delta^i h(b+k+1) = 0, \qquad 0 \leqslant i \leqslant n-k-1.$$

If $\lambda_0 \neq 0$ is an eigenvalue of M and $z_0(t)$ is a corresponding eigenvector, then $Mz_0(t) = \lambda_0 z_0(t)$ and

$$\lambda_0 (-1)^{n-k} Lz_0(t) = P(t)\, z_0(t+k)$$

or

$$(-1)^{n-k} Lz_0(t) = \frac{1}{\lambda_0} P(t)\, z_0(t+k)$$

and z_0 satisfies the boundary conditions (3). This is summarized in the following remark.

Remark 1. $\lambda_0 \neq 0$ is an eigenvalue of M with corresponding eigenfunction $z_0(t)$ iff $1/\lambda_0$ is an eigenvalue of (1), (3), with corresponding eigenfunction $z_0(t)$. Similar statements hold for the operator N and the eigenvalue problem (2), (3).

THEOREM 5. *In addition to* (H), *assume* $Q(t)\mathscr{K} \subseteq \mathscr{K}$ *for* $a \leqslant t \leqslant b$, *and for each nontrivial* $u \in \mathscr{P}$ *there is a* $t_u \in [a, b]$ *such that* $Q(t_u)\, u(t_u + k) \in \mathscr{K}^0$ (*interior of* $\mathscr{K}$). *Then the boundary value problem* (2), (3) *has a smallest positive eigenvalue* Λ_0 *and* Λ_0 *is smaller than the modulus of any other eigenvalue of* (2), (3). *Furthermore, there is an essentially unique eigenfunction* $z_0(t)$ *corresponding to* Λ_0 *and either* $z_0 \in \mathscr{P}^0$ *or* $-z_0 \in \mathscr{P}^0$.

Proof. We will show that $N: \mathscr{P}\backslash\{0\} \to \mathscr{P}^0$. To this end, let $0 \neq u \in \mathscr{P}$ and set

$$h(t) = Nu(t) = \sum_{s=a}^{b} G(t, s)\, Q(s)\, u(s+k).$$

It follows that h satisfies the boundary conditions (3). Further, it is easy to see that $h(t) \in \mathscr{K}$ for all $t \in [a+k, b+k]$. By hypothesis, there is a $t_u \in [a, b]$ such that $Q(t_u)\, u(t_u + k) \in \mathscr{K}^0$. By Theorem 1, $G(t, s) > 0$ for $a + k \leqslant t \leqslant b + k$. Hence

$$G(t, t_u)\, Q(t_u)\, u(t_u + k) \in \mathscr{K}^0.$$

It follows that $h(t) \in \mathscr{K}^0$, $a+k \leqslant t \leqslant b+k$, and from this it follows that $h \in \mathscr{P}^0$. Using standard arguments (for example, see [8, p. 253]), we now show that N is u_0-positive.

Since $N: \mathscr{P} \backslash \{0\} \to \mathscr{P}^0$, $\mathscr{P}^0 \neq \varnothing$. Let $u_0 \in \mathscr{P}^0$ and let $0 \neq u \in \mathscr{P}$. Since $u_0 \in \mathscr{P}^0$ and $Nu \in \mathscr{P}^0$, we can pick numbers k_2 sufficiently large and $k_1 > 0$ sufficiently small so that $u_0 - (1/k_2)\, Nu \in \mathscr{P}$ and $Nu - k_1 u_0 \in \mathscr{P}$. It follows that

$$k_1 u_0 \leqslant Nu \leqslant k_2 u_0$$

with respect to $\mathscr{P}$ and so N is u_0-positive. The conclusion of the theorem follows from Theorem 3 and Remark 1.

We now apply this result to the case where the cone $\mathscr{K}$ is a "quadrant" in R^m. Assume $\delta_i \in \{-1, 1\}$, $1 \leqslant i \leqslant m$, and define the "quadrant"

$$\mathscr{K}_1 = \{x \in R^m \colon \delta_i x_i \geqslant 0,\ 0 \leqslant i \leqslant m\}.$$

Then define the cone $\mathscr{P}_1$ in $\mathscr{B}$ by

$$\mathscr{P}_1 = \{u \in \mathscr{B} \colon u(t) \in \mathscr{K}_1,\ a+k \leqslant t \leqslant b+k\}.$$

COROLLARY 1. *If* (H) *holds, and* $\delta_i \delta_j q_{ij}(t) > 0$, $t \in [a, b]$, $1 \leqslant i, j \leqslant m$, *then the boundary value problem* (2), (3) *has a smallest positive eigenvalue* Λ_0 *which is smaller than the modulus of any other eigenvalue of* (2), (3). *Furthermore, there is an essentially unique eigenfunction* $z_0(t)$ *corresponding to* Λ_0 *and either* $z_0 \in \mathscr{P}_1^0$ *or* $-z_0 \in \mathscr{P}_1^0$.

Proof. Let $\mathscr{K} = \mathscr{K}_1$ and $\mathscr{P} = \mathscr{P}_1$ in Theorem 5. It suffices to show that $Q(t)\, \mathscr{K}_1 \subseteq \mathscr{K}_1$, $a \leqslant t \leqslant b$, and that for each $0 \neq u \in \mathscr{P}_1$ there is a $t_u \in [a, b]$ such that $Q(t_u)\, u(t_u + k) \in \mathscr{K}_1^0$.

Let $x \in \mathscr{K}_1$. Then $\delta_i x_i \geqslant 0$, $1 \leqslant i \leqslant m$. Then the ith component $(Q(t)x)_i$ satisfies

$$\begin{aligned} \delta_i (Q(t)\, x)_i &= \delta_i \sum_{j=1}^{m} q_{ij}(t)\, x_j \\ &= \sum_{j=1}^{m} \delta_i \delta_j q_{ij}(t)\, \delta_j x_j \geqslant 0 \end{aligned}$$

for $1 \leqslant i \leqslant m$, $a \leqslant t \leqslant b$. It follows that $Q(t)\, \mathscr{K}_1 \subseteq \mathscr{K}_1$ for $a \leqslant t \leqslant b$. Now assume $0 \neq u \in \mathscr{P}_1$. It follows that there is a $j_0 \in \{1, ..., m\}$ and a $t_u \in [a, b]$ such that $\delta_{j_0} u_{j_0}(t_u + k) > 0$. But then

$$\begin{aligned} \delta_i (Q(t_u)\, u(t_u + k))_i &= \sum_{j=1}^{m} \delta_i \delta_j q_{ij}(t_u)\, \delta_j u_j(t_u + k) \\ &\geqslant \delta_i \delta_{j_0} q_{ij_0}(t_u)\, \delta_{j_0} u_{j_0}(t_u + k) \\ &> 0 \end{aligned}$$

for $1 \leqslant i \leqslant m$. Hence $Q(t_u)\, u(t_u + k) \in \mathscr{K}_1^0$ and the result follows from Theorem 5.

THEOREM 6. *In addition to* (H), *assume* $P(t)$ *and* $Q(t)$ *satisfy the assumptions concerning* $Q(t)$ *in Theorem* 5. *If* $P(t) \leqslant Q(t)$ *with respect to* $\mathscr{K}$, $t \in [a, b]$, *then the smallest positive eigenvalues* λ_0 *and* Λ_0 *of* (1), (3) *and* (2), (3), *respectively, satisfy* $\Lambda_0 \leqslant \lambda_0$. *Furthermore, if* $\Lambda_0 = \lambda_0$ *then*

$$P(t)\, z_0(t+k) = Q(t)\, z_0(t+k), \qquad t \in [a, b],$$

where $z_0(t)$ *is as in Theorem* 5.

Proof. By Theorem 5, $\lambda_0 > 0$ and $\Lambda_0 > 0$ exist. We now show that $M \leqslant N$ with respect to $\mathscr{P}$. Let $u \in \mathscr{P}$ and note that

$$\begin{aligned} Mu(t) &= \sum_{s=a}^{b} G(t, s)\, P(s)\, u(s+k) \\ &\leqslant \sum_{s=a}^{b} G(t, s)\, Q(s)\, u(s+k) \\ &= Nu(t), \qquad t \in [a, b+n]. \end{aligned}$$

Further $\Delta^i Mu(a) = \Delta^i Nu(a) = 0$, $0 \leqslant i \leqslant k-1$, and $\Delta^i Mu(b+k+1) = \Delta^i Nu(b+k+1) = 0$, $0 \leqslant i \leqslant n-k-1$. Theorem 4 shows that $\Lambda_0 \leqslant \lambda_0$.

Now suppose $\Lambda_0 = \lambda_0$. By Theorem 4, the eigenfunctions $u(t)$, $v(t)$ of (1), (3) and (2), (3), respectively, are scalar multiples of each other, say $v(t) = cu(t)$. It follows that

$$(-1)^{n-k}\, Lv(t) = \lambda_0 P(t)\, v(t+k) = \lambda_0 Q(t)\, v(t+k), \qquad t \in [a, b].$$

Hence

$$P(t)\, z_0(t+k) = Q(t)\, z_0(t+k), \qquad t \in [a, b],$$

where $z_0(t) = v(t)$.

THEOREM 7. *Assume* $\delta_i \delta_j p_{ij}(t) \geqslant 0$ *on* $[a, b]$ *for* $1 \leqslant i, j \leqslant m$, *and that there is a* $t_0 \in [a, b]$ *and an* $i_0 \in \{1, ..., m\}$ *such that* $p_{i_0 i_0}(t_0) > 0$. *Then the eigenvalue problem* (1), (3) *has a least positive eigenvalue* λ_0 *which is a lower bound on the modulus of the eigenvalues of* (1), (3) *and satisfies*

$$\lambda_0^{-1} \geqslant G(t_0 + k, t_0)\, p_{i_0 i_0}(t_0).$$

Furthermore, there is an eigenfunction $y_0(t)$ *corresponding to* λ_0 *satisfying* $\delta_i (y_0(t))_i \geqslant 0$, $t \in [a, b+n]$, *for* $1 \leqslant i \leqslant m$.

Proof. First we show that $M: \mathscr{P}_1 \to \mathscr{P}_1$, where

$$Mu(t) = \sum_{s=a}^{b} G(t, s)\, P(s)\, u(s+k).$$

Let $u \in \mathscr{P}_1$ and consider

$$\begin{aligned} \delta_i (Mu)_i\,(t) &= \sum_{s=a}^{b} G(t, s) \sum_{j=1}^{m} \delta_i \delta_j\, p_{ij}(s)\, \delta_j u_j(s+k) \\ &\geqslant 0, \qquad 1 \leqslant i \leqslant m,\ t \in [a, b+n]. \end{aligned}$$

Further, $Mu(t)$ satisfies the boundary conditions (3). Hence, $M: \mathscr{P}_1 \to \mathscr{P}_1$.

Define $w \in \mathscr{P}_1$ by setting $w_i(t) = 0$ on $[a, b+n]$ for $i \neq i_0$, and set

$$w_{i_0}(t) = \begin{cases} 0, & t \neq t_0 + k \\ \delta_{i_0}, & t = t_0 + k, \end{cases}$$

where i_0 and t_0 are as in the statement of the theorem. Note that

$$\varepsilon_0 \equiv G(t_0 + k, t_0)\, p_{i_0 i_0}(t_0) > 0.$$

Then for $i \neq i_0$ we have

$$\delta_i (Mw)_i(t) \geqslant 0 = \varepsilon_0 \delta_i w_i(t), \qquad t \in [a, b+n].$$

Further, for $t \neq t_0 + k$,

$$\delta_{i_0} (Mw)_{i_0}(t) \geqslant 0 = \varepsilon_0 \delta_{i_0} w_{i_0}(t).$$

We also have that

$$\begin{aligned} \delta_{i_0} (Mw)_{i_0}\,(t_0 + k) &= \sum_{s=a}^{b} G(t_0 + k, s) \sum_{j=1}^{m} \delta_{i_0} \delta_j\, p_{i_0 j}(s)\, \delta_j w_j(s+k) \\ &= G(t_0 + k, t_0)\, p_{i_0 i_0}(t_0)\, \delta_{i_0} w_{i_0}(t_0 + k) \\ &= \varepsilon_0 \delta_{i_0} w_{i_0}(t_0 + k). \end{aligned}$$

It follows that $Mw \geqslant \varepsilon_0 w$ with respect to $\mathscr{P}_1$. The conclusions of this theorem now follow easily from Theorem 2.

By finding the appropriate Green's function, it is easy to get the following result.

COROLLARY 2. *If $P(t)$ satisfies the hypothesis of Theorem 7, then the eigenvalue problem*

$$-\Delta^2 y(t)=\lambda P(t)\, y(t+1)$$
$$y(a)=0$$
$$y(b+2)=0$$

has a smallest positive eigenvalue λ_0 which satisfies

$$\lambda_0^{-1}\geqslant\frac{(t_0+1-a)(b+1-t_0)}{b+2-a}\,p_{i_0 i_0}(t_0).$$

In Theorem 7, we obtained an upper bound for λ_0. Using a proof similar to a proof of Ahmad and Lazer [1, Lemma 1] in the differential equations case, we can also get a lower bound for λ_0.

COROLLARY 3. *Assume $P(t)$ satisfies the hypothesis of Theorem 7. Then the least positive eigenvalue λ_0 of (1), (3) satisfies*

$$G(t_0+k,t_0)\,p_{i_0 i_0}(t_0)\leqslant\lambda_0^{-1}\leqslant B\sum_{s=a}^{b}\|P(s)\|,$$

where $B=\max\{G(t,s)\mid t\in[a+k,b+k],\ \ s\in[a,b]\}$ *and* $\|P(s)\|=\max_{1\leqslant i\leqslant m}\sum_{j=1}^{m}\delta_i\delta_j p_{ij}(s)$.

Proof. Let λ_0 be the smallest positive eigenvalue and let $z_0(t)$ be a corresponding eigenvector in $\mathscr{P}_1$. Pick $t_0\in[a,b]$ and $j_0\in\{1,...,m\}$ such that

$$A\equiv\delta_{j_0}(z_0(t_0+k))_{j_0}=\max\{\delta_j(z_0(t+k))_j\mid 1\leqslant j\leqslant m,\ t\in[a,b]\}.$$

Then $Mz_0(t)=(1/\lambda_0)\,z_0(t)$, or equivalently,

$$\frac{1}{\lambda_0}\delta_{j_0}(z_0(t_0+k))_{j_0}=\sum_{s=a}^{b}G(t_0+k,s)\sum_{j=1}^{m}\delta_{j_0}\delta_j p_{j_0 j}(s)\,\delta_j(z_0(s+k))_j.$$

This implies that

$$\frac{1}{\lambda_0}A\leqslant BA\sum_{s=a}^{b}\sum_{j=1}^{m}\delta_{j_0}\delta_j p_{j_0 j}(s).$$

It follows that

$$\lambda_0^{-1}\leqslant B\sum_{s=a}^{b}\|P(s)\|.$$

THEOREM 8. *In addition to* (H), *assume*

1. *there is an* $i_0 \in \{1, ..., m\}$ *and a* $t_0 \in [a, b]$ *such that* $p_{i_0 i_0}(t_0) > 0$, *and*
2. $0 \leqslant p_{ij}(t)\, \delta_i \delta_j \leqslant q_{ij}(t)\, \delta_i \delta_j$ *and* $q_{ij}(t) \neq 0$ *on* $[a, b]$ *for* $1 \leqslant i, j \leqslant m$.

Then the eigenvalue problems (1), (3) *and* (2), (3) *have smallest positive eigenvalues* λ_0 *and* Λ_0, *respectively. Furthermore,* $\Lambda_0 \leqslant \lambda_0$ *and* $\Lambda_0 = \lambda_0$ *iff* $P(t) = Q(t)$ *on* $[a, b]$.

Proof. By Corollary 1 and Theorem 7, it follows that Λ_0 and λ_0 exist. The proof of Theorem 6 still applies in the present context, since only one of the operators M, N is required to be u_0-positive in that proof. Hence, $\Lambda_0 \leqslant \lambda_0$.

Assume now that $\Lambda_0 = \lambda_0$. By Corollary 1, there is an eigenfunction $z_0(t) \in \mathscr{P}_1^0$, and the arguments in Theorem 6 show that

$$P(t)\, z_0(t+k) = Q(t)\, z_0(t+k), \qquad t \in [a, b].$$

It follows that for $t \in [a, b]$,

$$\sum_{j=1}^{m} \delta_i \delta_j [q_{ij}(t) - p_{ij}(t)]\, \delta_j (z_0(t+k))_j = 0.$$

Since every term in this sum is nonnegative and $\delta_j\ (z_0(t))_j > 0$ for $t \in [a+k, b+k]$, $1 \leqslant j \leqslant m$, we see that

$$p_{ij}(t) = q_{ij}(t), \qquad t \in [a, b],\ 1 \leqslant i, j \leqslant m.$$

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